Session 8 : Solutions

The Ising model in 1D is $H = -J\sum_i s_i s_{i+1} - h\sum_i s_i$

$$Z = \sum_{\{s_i\}} e^{\beta J \sum_i s_i s_{i+1} - h\sum_i s_i}$$

Let's coarse grain : we do not touch the even sites, and we sum over the odd ones

$$Z = \sum_{\{s_i\}_{i\,\text{even}}} e^{\beta h \sum_{i=-\infty}^{+\infty} s_{2i}} \sum_{\{s_i\}_{i\,\text{odd}}} e^{\beta h \sum_{i=-\infty}^{+\infty} s_{2i+1} + \beta J \cdot \underbrace{\sum_{i=-\infty}^{+\infty} s_{2i+1}(s_{2i}+s_{2i+2})}_{\text{I symmetrize it}}} =$$

$$= \sum_{\{s_i\}_{i\,\text{even}}} e^{\beta h \sum_{i=-\infty}^{+\infty} s_{2i}} \sum_{\{s_i\}_{i\,\text{odd}}} \prod_{i=-\infty}^{+\infty} e^{s_{2i+1}\left[\beta h + \beta J\cdot(s_{2i}+s_{2i+2})\right]} =$$

$$= \sum_{\{s_i\}_{i\,\text{even}}} e^{\beta h \sum_{i=-\infty}^{+\infty} s_{2i}} \prod_{i=-\infty}^{+\infty} \left[2\cosh\left[\beta h + \beta J\,(s_{2i}+s_{2i+2})\right]\right]$$

$$= \sum_{\{s_i\}_{i\,\text{even}}} e^{\beta \frac{h}{2} \sum_{i=-\infty}^{+\infty} (s_{2i}+s_{2i+2})} \prod_{i=-\infty}^{+\infty} 2\cosh\left[\beta h + \beta J\,(s_{2i}+s_{2i+2})\right] =$$

$$= \sum_{\{s_i\}_{i\,\text{even}}} \prod_{i=-\infty}^{+\infty} 2 e^{\beta \frac{h}{2}(s_{2i}+s_{2i+2})} \cosh\left[\beta h + \beta J\,(s_{2i}+s_{2i+2})\right] =$$

$$= \sum_{\{s_i\}_{i\,\text{even}}} \prod_{i=-\infty}^{+\infty} 2\, e^{\frac{1}{2}K_1(s_{2i}+s_{2i+2})} \cosh\left[K_1 + K_2\,(s_{2i}+s_{2i+2})\right]$$

$$K_1 = \beta h \qquad K_2 = \beta J$$

How does this compare with the original Ising model before coarse graining ?

$$Z = \sum_{\{s_i\}} e^{\beta h \sum_i s_i + \beta J \sum_i s_i s_{i+1}} =$$

$$= \sum_{\{s_i\}} \prod_i e^{\frac{K_1}{2}(s_i + s_{i+1}) + K_2 s_i s_{i+1}}$$

If we want to write the coarse-grained expression as a new Ising model, we have to match

there is no problem adding it

$$e^{\frac{K_1'}{2}(s_i + s_{i+1}) + K_2' s_i s_{i+1} + K_0'} = 2\, e^{\frac{K_1}{2}(s_i + s_{i+1}) + K_0} \cdot \cosh\left[K_1 + K_2 \cdot (s_i + s_{i+1})\right]$$

Here we have changed $2i \to i$ since only the even sites have remained

We can match thes conditions case by case

$$\begin{array}{ll} + + & e^{K_1' + K_2' + K_0'} = 2\, e^{K_0 + K_1} \cosh\left[K_1 + 2K_2\right] \\ \begin{array}{l} + - \\ - + \end{array} & e^{-K_2' + K_0'} = 2\, e^{K_0} \cosh\left[K_1\right] \\ - - & e^{-K_1' + K_2' + K_0'} = 2\, e^{K_0 - K_1} \cosh\left[K_1 - 2K_2\right] \end{array}$$

From here we can start solving the equations

- divide the first by the third:

$$e^{2K_1'} = e^{2K_1} \frac{\cosh(K_1 + 2K_2)}{\cosh(K_1 - 2K_2)}$$

- multiply (1) and (3) and divide by $(2)^2$:

$$e^{4K_2'} = \frac{\cosh(K_1+2K_2)\cosh(K_1-2K_2)}{\cosh(K_1)^2}$$

- and at last

$$e^{4K_0'} = 8\, e^{K_0} \cosh(2K_2 + K_1)\cosh(2K_2 - K_1)\cosh^2(K_1)$$

Now we have the iterative map

$$e^{2K_1'} = e^{2K_1} \frac{\cosh(K_1 + 2K_2)}{\cosh(K_1 - 2K_2)}$$

$$e^{4K_2'} = \frac{\cosh(K_1+2K_2)\cosh(K_1-2K_2)}{\cosh(K_1)^2}$$

Let's now focus on $K_1 = 0$ ($h = 0$: no field)

The two equations become

$$\begin{cases} e^{2K_1'} = 1 & \Longrightarrow K_1' = 0 \\ e^{4K_2'} = \cosh^2(2K_2) \end{cases}$$

Does $K_1' = 0$? K_1' multiplies s_i : its contribution to the energy depends on the state of s_i. It breaks the $\mathbb{Z}_2$ symmetry $(+1 \leftrightarrow -1)$

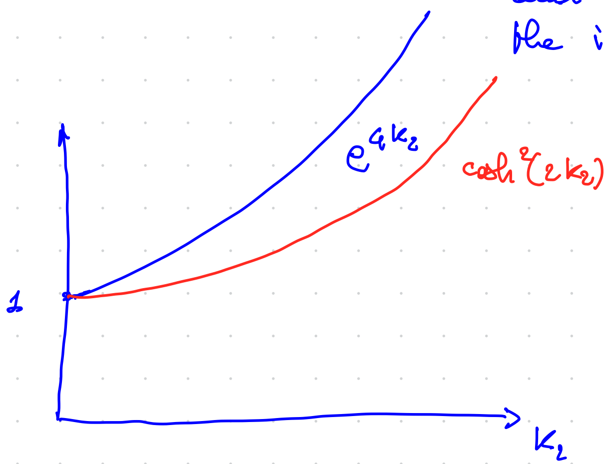
If $h = 0$ the $\mathbb{Z}_2$ symmetry is true, and it must be preserved by the coarse-graining

$\Longrightarrow K_1' = 0$ is the correct answer

Let's look at the other equation:

$$e^{4K_2'} = \cosh^2(2K_2)$$

and let's solve it graphically (when solving graphically, $K' = K$ in the graph and we look for the intersections)



Where do they intersect?

$K_2 = 0 \implies \beta J = 0 \implies \frac{J}{k_B T} = 0 \implies T = \infty$

$K_2 = \infty \implies \beta J = \infty \implies \frac{J}{k_B T} = \infty \implies T = 0$

There are no fixed points for finite K_2 $\implies$ no critical point!

This is correct for the 1D Ising model!

What about the stability?

Let's focus on $K_2^* = 0$ $\quad (T = \infty)$

$$K_2 = K_2^* + \delta K_2 = \delta K_2$$

$$K_2' = K_2^* + \delta K_2' = \delta K_2'$$

$$e^{4\delta K_2'} = \cosh^2(2\delta K_2)$$

$\downarrow$ linear expand

$$1 + 4\delta K_2' = (1 + \delta K_2)^2 = 1 + 2\delta K_2$$

$$\Rightarrow \quad \delta K_2' = \frac{1}{2}\delta K_2$$

It is a contraction! $K_2^* = 0$ is stable

Since the only one left is $K_2^* = \infty$, it must be unstable. The renormalization flow is